ST790 – Homework 2 Due: 11/22/2022

These exercises are meant to supplement the lectures by providing some further examples to illustrate the general ideas and theory. Students (individually or in pairs) should attempt to solve all the assigned problems. The solutions will be collected at the end of the semester (the day before Thanksgiving), so you may work on these at your own pace. But don't wait too long to get started! If you have questions, feel free to ask.

1. Suppose that the functional $\psi : 2^{\mathbb{X}} \to [0,1]$ is K-monotone. Show that it's also J-monotone for all non-negative integers $J \leq K$.

2. Let X be a random vector in $\mathbb{X} \subseteq \mathbb{R}^q$ and define $\mathcal{X} = \{x \in \mathbb{X} : \|x - X\| \leq r\}$, where $\|\cdot\|$ is the Euclidean norm and $r > 0$ is a fixed radius. Prove that the events $\mathcal{X} \cap K \neq \varnothing$ are measurable for all compact $K \subset \mathbb{X}$, so that $\mathcal{X}$ is a random set.

   Hints: The events $X \in B$ are measurable for all closed B; for a given compact K, you'll need the *fattening* K_r as $\{x \in \mathbb{X} : \inf_{y \in K} \|x - y\| \leq r\}$.

3. Let X be a random vector in $\mathbb{X} \subseteq \mathbb{R}^q$ and $g : \mathbb{X} \to \mathbb{R}$ a suitable function. Define

$$\mathcal{X} = \{x \in \mathbb{X} : g(X) \leq g(x)\}.$$

   (a) Prove that, if g is continuous, then $\mathcal{X}$ is a random set by showing that the event $\mathcal{X} \cap K \neq \varnothing$ is measurable for all compact K.
   Hint: Show that $\mathcal{X} \cap K \neq \varnothing$ iff $g(X) \leq \sup_{x \in K} g(x)$.

   (b) Take $X \sim \mathsf{N}(0,1)$ and $g(x) = e^{-|x-1|}$ for $x \in \mathbb{X} = \mathbb{R}$, and let $\mathcal{X}$ be the random set as defined above. Recall the hitting probability

$$\pi(x) = \mathsf{P}(\mathcal{X} \ni x), \quad x \in \mathbb{X}.$$

      i. Without evaluating the expression, explain why $\pi(1) = 1$.
      ii. Draw a plot of the function π and, in particular, confirm that $\pi(1) = 1$. You can get a more-or-less closed-form expression for π, but I'd prefer that you evaluate it via Monte Carlo, just for the experience. Maybe try it both ways—analytically and numerically—and compare results.

   (c) Let X be a continuous random vector and define the random variable $Y = \pi(X)$, where π is the random set $\mathcal{X}$'s hitting probability. What additional condition on g is needed to ensure that $Y \sim \mathsf{Unif}(0,1)$?
   Hint: The random variable $g(X)$ might not be continuous even if both X and g are continuous—what can make continuity fail?

4. Let $\mathcal{X}$ be a random set. Show that, for all subsets $A \subseteq \mathbb{X}$, the functions $\underline{\Pi}(A) = \mathsf{P}(\mathcal{X} \subseteq A)$ and $\overline{\Pi}(A) = \mathsf{P}(\mathcal{X} \cap A \neq \varnothing)$ are dual/conjugates of one another.

5. Let $\overline{\Pi}(A) = \mathsf{P}(\mathcal{X} \cap A \neq \varnothing)$ for a random set $\mathcal{X}$. Show that $\overline{\Pi}$ is 2-alternating, i.e.,

$$\overline{\Pi}(A \cup B) \leq \overline{\Pi}(A) + \overline{\Pi}(B) - \overline{\Pi}(A \cap B), \quad \text{all } A \text{ and } B.$$

   Hint: Convince yourself that

$$\mathsf{P}(\mathcal{X} \cap A \neq \varnothing \text{ and } \mathcal{X} \cap B \neq \varnothing) \geq \mathsf{P}\{\mathcal{X} \cap (A \cap B) \neq \varnothing\}.$$

6. Consider the case of finite $\mathbb{X}$. The goal here is to prove the *Möbius inversion* results presented in lecture, that is, if $f(A) = \mathsf{P}(\mathcal{X} = A)$ is the mass function of $\mathcal{X}$, and $F(A) = \mathsf{P}(\mathcal{X} \subseteq A)$ is the corresponding "distribution function" of $\mathcal{X}$, then the two are related by by the formulas

$$
\begin{aligned}
F(A) &= \sum_{B \in 2^{\mathbb{X}} : B \subseteq A} f(B) \\
f(A) &= \sum_{B \in 2^{\mathbb{X}} : B \subseteq A} (-1)^{|A \cap B^c|} F(B). \qquad (1)
\end{aligned}
$$

Recall that $|A|$ denotes the cardinality of the finite set A, i.e., the number of elements in A. The first relation is a straightforward application of the rules of probability; it's the latter relation that this exercise is focused on establishing. The parts below walk you through the basic combinatorial arguments step by step.[1]

(a) Let $A = \{x_1, \ldots, x_K\}$ be a finite set with $K \geq 0$ elements. Show that

$$
\sum_{B \in 2^{\mathbb{X}} : B \subseteq A} (-1)^{|B|} = \begin{cases} 0 & \text{if } A \neq \varnothing, \text{ i.e., } K > 0 \\ 1 & \text{if } A = \varnothing, \text{ i.e., } K = 0. \end{cases}
$$

Hints: If $K > 0$, then $(1-1)^K = 0$; expand $(1-1)^K$ using the binomial theorem; if $K > 0$ and $k \leq K$, then how many subsets B of A have $|B| = k$?

(b) Let A and B be finite sets with $B \subseteq A$. Show that

$$
\sum_{C \in 2^{\mathbb{X}} : B \subseteq C \subseteq A} (-1)^{|C|} = \begin{cases} (-1)^{|A|} & \text{if } B = A \\ 0 & \text{otherwise.} \end{cases}
$$

Hints: Make a "change-of-variables" $D = C \cap B^c$, i.e., everything in C that's not in B, so that D ranges over subsets of $A \cap B^c$; simplify the summation and apply the result in Part (a).

(c) Prove the result in Equation (1).

Hints: Start with the right-hand side of (1), simplify a bit, then plug in the definition of $F(\cdot)$ in terms of $f(\cdot)$, i.e.,

$$
\begin{aligned}
\sum_{B \in 2^{\mathbb{X}} : B \subseteq A} (-1)^{|A \cap B^c|} F(B) &= (-1)^{|A|} \sum_{B \in 2^{\mathbb{X}} : B \subseteq A} (-1)^{|B|} F(B) \\
&= (-1)^{|A|} \sum_{B \in 2^{\mathbb{X}} : B \subseteq A} (-1)^{|B|} \sum_{C \in 2^{\mathbb{X}} : C \subseteq B} f(C);
\end{aligned}
$$

now interchange the order of the two finite sums (so that B ranges between C and A), simplify a bit, and apply the result in Part (b) above.

7. The three-state universe $\mathbb{X} = \{x_1, x_2, x_3\}$ in Exercise 6 of Homework 1 had lower and upper probabilities given in the following table:

[1] See Chapter 2.7 in Shafer's *A Mathematical Theory of Evidence*, 1976.

	x_1	x_2	x_3
lower probability	0.45	0.15	0.30
upper probability	0.55	0.20	0.40

You sketched out the corresponding credal set based on these, so you know it's non-empty and, therefore, we're dealing with a capacity so the lower probability is at least 2-monotone, but maybe more. The claim is that this capacity is induces by a finite random set $\mathcal{X}$, which means ∞-monotonicity. You'll do this by checking that there is a corresponding mass function on $2^{\mathbb{X}}$.

(a) Find the lower probability for all $2^3 = 8$ subsets of $\mathbb{X}$ and arrange it in a "A vs. $F(A)$" table like in lecture.

(b) Find the Möbius inverse of F and confirm that it's a mass function on $2^{\mathbb{X}}$.

8. Let $h : \mathbb{X} \to [0, 1]$ be continuous (or upper semicontinuous) and be such that $\sup_{x \in \mathbb{X}} h(x) = 1$. Define the random sets

$$\mathcal{X}_j = \{x \in \mathbb{X} : h(x) \geq U_j\}, \quad j = 1, 2,$$

where $U_1, U_2 \overset{\text{iid}}{\sim} \mathsf{Unif}(0, 1)$. Both $\mathcal{X}_1$ and $\mathcal{X}_2$ induce capacities that are maxitive.

(a) Define the new random set $\mathcal{X} = \mathcal{X}_1 \cap \mathcal{X}_2$. Find the hitting probability function of $\mathcal{X}$ and show that it induces a capacity that's maxitive.

(b) Choose a suitable and sufficiently function h (could be one of the examples from lecture) and plot it and the hitting probability from Part (a) overlaid on the same figure. What does this plot reveals about the structure of a typical realization of $\mathcal{X}$ compared to that of its components $\mathcal{X}_1$ and $\mathcal{X}_2$?

Hint: The structure I'm interested in here is obvious from the definition of $\mathcal{X}$, but I want you to see how what is obvious is confirmed by the plot.

(c) Argue that $\mathsf{P}(\mathcal{X} = \varnothing) = 0$. It's not generally true, however, that the intersection of two random sets is non-empty with probability 1, even if the individual random are. What's special about the $\mathcal{X}_1$ and $\mathcal{X}_2$ considered here that ensures the intersection is non-empty.