ST790 – Homework 3 **Due: 11/22/2022**

These exercises are meant to supplement the lectures by providing some further examples to illustrate the general ideas and theory. Students (individually or in pairs) should attempt to solve all the assigned problems. The solutions will be collected at the end of the semester (the day before Thanksgiving), so you may work on these at your own pace. But don't wait too long to get started! If you have questions, feel free to ask.

1. Let $\overline{\Pi}$ be an upper probability (i.e., a capacity) on $\mathbb{X}$ that's not necessarily a possibility measure. Define its contour $\pi(x) := \overline{\Pi}(\{x\})$, $x \in \mathbb{X}$, which is just the upper probability of the singleton set $\{x\}$. Show that $\sup_{x \in A} \pi(x) \leq \overline{\Pi}(A)$ for all A. So, if $\overline{\Pi}$ happens to be a possibility measure, then the relationship between the magnitudes of π and $\overline{\Pi}$ are "extreme" in a certain sense.

2. By now we've seen several different indirect arguments that possibility measures avoid sure loss. But it's instructive to see how this can be shown directly by constructing a probability distribution that's in the given possibility measure's credal set. Such an argument is presented in Dominik Hose's PhD thesis[1] (bottom of page 30 through the top of page 31). Rewrite Dominik's proof in your own words, making sure all the steps are clear.

3. Refer to page 10 in the Week 03b slides. There we have a given possibility measure $\overline{\Pi}$, with contour π, and asking which probabilities P are contained in the credal set $\mathscr{C}(\overline{\Pi})$. A claim is made that one of the elements in the credal set is $\mathsf{P}^\star$, where

$$\mathsf{P}^\star(A) = \int_0^1 \frac{\mathsf{P}_0(S_\alpha \cap A)}{\mathsf{P}_0(S_\alpha)} \, d\alpha, \quad A \subseteq \mathbb{X},$$

   with $S_\alpha = \{x : \pi(x) > \alpha\}$ the α-super-level set of π and P_0 a probability that assigns positive mass to all of the super-level sets. Prove this claim.

   Hint: Recall that there's a characterization of the contents of the credal set $\mathscr{C}(\overline{\Pi})$. So, all you need to show is that $\mathsf{P}^\star(S_\beta) \geq 1 - \beta$ for all $\beta \in [0, 1]$.

4. For the probability-to-possibility transform discussed in Week 04a, show or argue that only the level sets of the function f are relevant. It's fine if you just reason this out and/or focus on a particular example. The take-away message is that there are distinct f that produce the same π_f.

5. Consider two random sets, $\mathcal{X}_1$ and $\mathcal{X}_2$, where

$$\mathcal{X}_i = \{x \in \mathbb{X} : \|x - X_i\| \leq r_i\}, \quad i = 1, 2,$$

   with $\mathbb{X} = \mathbb{R}^q$ for some $q \geq 1$, $\| \cdot \|$ the usual Euclidean norm on $\mathbb{R}^q$, r_1 and r_2 are two fixed positive numbers, and X_1 and X_2 are iid.

   (a) Write an expression for $\mathsf{P}(\mathcal{X}_1 \cap \mathcal{X}_2 \neq \varnothing)$.

[1] The thesis is available to NC State students on the course Moodle page; anyone else who wants access to the thesis, please let me know and I'll send it to you.

(b) Find a special case of the general setup above where you can simplify the expression in Part (a) and then do the simplification.

(c) Any interesting plots you can think of drawing to visualize how the probability in Part (a) depends on certain inputs? For example, that probability would depend on (r_1, r_2) and on features of the common distribution of X_1, X_2.

6. Define the random sets $\mathcal{X}_i = \{x \in \mathbb{X} : |x| \leq |X_i|\}$, for $i = 1, 2$, where X_1 and X_2 are iid $\mathsf{N}(0, 1)$. Let $A_y = (-\infty, y]$ denote the half-line, for $y \in \mathbb{R}$.

(a) Define the "lower" and "upper distribution functions"

$$\underline{G}_1(y) = \mathsf{P}(\mathcal{X}_1 \subseteq A_y) \quad \text{and} \quad \overline{G}_1(y) = \mathsf{P}(\mathcal{X}_1 \cap A_y \neq \varnothing).$$

Draw the graphs of these two functions on the same plot. Are the names "lower" and "upper distribution functions" justified?

(b) Show that $\mathsf{P}(\mathcal{X}_1 \cap \mathcal{X}_2 \neq \varnothing) = 0$

(c) Define the new "lower" and "upper distribution functions"

$$\underline{G}_{12}(y) = \mathsf{P}(\mathcal{X}_1 \cap \mathcal{X}_2 \subseteq A_y) \quad \text{and} \quad \overline{G}_{12}(y) = \mathsf{P}(\mathcal{X}_1 \cap \mathcal{X}_2 \cap A_y \neq \varnothing).$$

(This is the result of applying Dempster's rule of combination to the belief functions determined by $\mathcal{X}_1$ and $\mathcal{X}_2$.) Draw the graphs of these two functions on the same plot. How do these compare to those in Part (b)?

7. Consider two simple support functions as defined on page 9 of the Week 04b slides. These belief functions, $\underline{\Pi}_i$, are determined by pairs (s_i, S_i), for $i = 1, 2$, consisting of a number $s_i \in [0, 1]$ and a subset S_i of $\mathbb{X}$.

(a) Argue that the mass function m_i corresponding to $\underline{\Pi}_i$ is given by

$$m_i(A) = \begin{cases} s_i & \text{if } A = S_i \\ 1 - s_i & \text{if } A = \mathbb{X} \\ 0 & \text{otherwise.} \end{cases}$$

(b) Define the degree of conflict κ as

$$\kappa = \sum_{(A_1, A_2): A_1 \cap A_2 = \varnothing} m_1(A_i) m_2(A_2).$$

Show/argue that, if $S_1 \cap S_2 \neq \varnothing$, then $\kappa = 0$.

(c) Find an expression for the mass function m_{12} of the belief function obtained by applying Dempster's rule of combination to m_1 and m_2.

Hint: The combined belief function only has four focal elements so you can just calculate the mass of each one.

8. Let $\pi(x) = e^{-|x|}$ be a possibility contour and $\overline{\Pi}$ the corresponding possibility measure (obtained via optimization).

(a) Let $f(x) = (x-1)^2$. Use the second[2] formula on page 13 of the Week 05b slides to find the Choquet integral of f with respect to $\overline{\Pi}$. That is, evaluate

$$\int_0^1 \sup\{f(x) : \pi(x) \geq \alpha\}\, d\alpha.$$

(b) Find $\mathsf{E}\{f(X)\}$ where $X \sim \mathsf{Laplace}(0,1)$, i.e., $X = |V|$ where $V \sim \mathsf{Exp}(1)$.

(c) Explain why it's not a coincidence that the answer to Part (a) is greater than the answer to (b).

Hint: If you can show that the Laplace distribution is in the credal set of $\overline{\Pi}$, then you're done, right?

(d) Repeat Parts (a) and (b) but with the function $g(x) = x^2$. Both calcuations give you the same numerical answer, any idea why? I don't expect you to know how to answer this, so it's OK if you don't.

[2] In the slides I gave two formulas and said they're "equivalent." This is true for *bounded* functions f but might not be true for unbounded f. Proposition 15.42 in Troffaes & De Cooman's *Lower Previsions* book gives a formula like the second one on my slides for certain unbounded f, ones they call *previsable*, a condition which is satisfied in this example.