ST790 — Fall 2022
Imprecise-Probabilistic Foundations of Statistics

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Week 02a

This lecture

- Random sets: definition and examples
- Distribution of random sets
- Properties of induced capacity
    - non-additive
    - 2- and even ∞-monotone/alternating
    - (semi-)continuity
- More examples
- .....

Random sets

- A *random set* is exactly what the name suggests:
  - it's a *set-valued* random variable,
  - or a random variable whose realizations are sets
- Not uncommon:
  - confidence regions
  - survey sampling
  - censored/missing/coarse data
  - *technically, every continuous data application*
- Question: *How to describe the distribution of a random set?*
- This random set distribution induces a *capacity*
- Special kind of imprecise probability
  - interesting in their own right
  - also a good entry point to IP

Applications[1]

- Sample surveys:
  - sampling plan designed to get a "representative sample"
  - output is a *set* of units
- Coarse data:
  - $\mathcal{X} = \{X\}$, exact observation
  - $\mathcal{X} = (-\infty, \infty)$, missing observation
  - $\mathcal{X} = [L, U]$, interval censoring
- Artificial intelligence:
  - humans process information without exact observations
  - e.g., {my friend is kinda far away}
  - data is basically a *set* of distances

[1] More-or-less from Nguyen's book

Examples[2]

1. *Random singleton.* If X is a random variable or vector, then $\mathcal{X} = \{X\}$ is a random set, a random singleton
2. *Random interval.* If X is a random vector and $a(\cdot) \leq b(\cdot)$ are functions, then $\mathcal{X} = [a(X), b(X)]$ is a random interval
   - one-sided: $(-\infty, b(X)]$ or $[a(X), \infty)$
   - random center, fixed width: $[c(X) - \alpha, c(X) + \beta]$
   - fixed center, random width: $c \pm |d(X) - c|$
3. *Random ball.* If X is a random vector and $R > 0$ is a random variable, then $\mathcal{X} = \{x : \|x - X\| \leq R\}$ is a random ball.

[2] From Molchanov's book

Examples, cont.

4. *Random triangle.* If X_1, X_2, and X_3 are random vectors, then $\mathcal{X} =$ convex hull of $\{X_1, X_2, X_3\}$ is a random triangle
5. *Random level sets.* If $\{X_t : t \in \mathbb{T}\}$ is a real-valued process w/ continuous sample paths, then, for each $x \in \mathbb{R}$,
   - $\mathcal{X} = \{t : X_t = x\}$ is a random level set
   - $\mathcal{X} = \{t : X_t \geq x\}$ is a random upper level set

Formal definition

- Probability space $(\Omega, \mathcal{F}, \mathrm{P})$, where $\mathcal{F}$ is a σ-algebra
- $\mathcal{F}$ contains events, subsets of outcomes we can "witness"[3]
- A random variable $X : \Omega \to \mathbb{R}$ is a measurable function, i.e.,

$$\{\omega : X(\omega) \leq \alpha\} \in \mathcal{F}, \quad \text{for all } \alpha \in \mathbb{R}$$

- Same idea for random sets
  - $\mathcal{X} : \Omega \to 2^{\mathbb{X}}$ is a *random set* if

$$\{\omega : \mathcal{X}(\omega) \cap K \neq \varnothing\} \in \mathcal{F}, \quad \text{for all compact } K \subseteq \mathbb{X}$$

  - That is, we can witness if a random set $\mathcal{X}$ intersects with all compact sets K in its range
- Implies relevant functionals $\phi(\mathcal{X})$ are random variables

[3] $\mathcal{F}$ is *huge*, but generally a sub-collection of the power set 2^{Ω}

Examples, cont.

2. If $\mathcal{X} = [a(X), b(X)]$, then

$$\mathcal{X} \cap K \neq \varnothing \iff a(X) \leq \sup K \text{ or } b(X) \geq \inf K$$

So, if a and b are measurable, then $\mathcal{X}$ is a random set

5. If $\mathcal{X} = \{t : X_t \geq x\}$ for fixed x, then

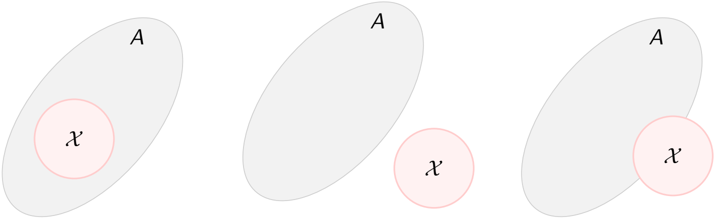
$$\mathcal{X} \cap K \neq \varnothing \iff \sup_{t \in K} X_t \geq x$$

If $t \mapsto X_t(\omega)$ is continuous, then $\omega \mapsto \sup_{t \in K} X_t(\omega)$ is measurable, hence so is the right-most event above

Take-away point: measurability of $\mathcal{X}$ typically boils down to simpler or more familiar questions about its components

Distribution of a random set

- Random variables: distribution function $x \mapsto \mathrm{P}(X \leq x)$
- Similar idea for random sets: $A \mapsto \mathrm{P}(\mathcal{X} \subseteq A)$
- But the above functional *is not a measure!*
- $\mathrm{P}(\mathcal{X} \subseteq A) + \mathrm{P}(\mathcal{X} \subseteq A^c) \leq 1$[4]



(a) $\mathcal{X} \subseteq A$ (b) $\mathcal{X} \subseteq A^c$ (c) Otherwise

[4] Strict inequality for some A unless $\mathcal{X}$ is a random singleton

Capacities

- In random set literature, the *capacity* associated with a random set $\mathcal{X}$ is usually taken to be

$$\overline{\Pi}(A) = \mathrm{P}(\mathcal{X} \cap A \neq \varnothing), \quad A \subseteq \mathbb{X}$$

- The dual $\underline{\Pi}(A) = \mathrm{P}(\mathcal{X} \subseteq A)$ is also a capacity
- Technical note:
  - previously, "$\mathcal{X} \cap K \neq \varnothing$" measurable only for *compact* K
  - but here we consider all A, not just compacts
  - we're dealing with the capacity's *extension*[5] to $2^{\mathbb{X}}$
- *Hitting probability* is a special case where A is a singleton:

$$\pi(x) = \mathrm{P}(\mathcal{X} \ni x), \quad x \in \mathbb{X}$$

[5] Theorem 1.12 in Molchanov's book. Compare this to how Lebesgue measure is initially defined for intervals and then "extended" to Borel sets

Capacities, cont.

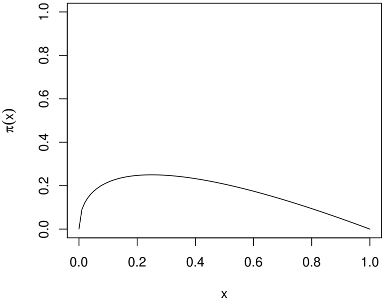
Example. $\mathcal{X} = [X^2, X]$ and $X \sim \text{Unif}(0,1)$

- For an interval $A = [a_1, a_2]$,

$$\begin{aligned}\overline{\Pi}(A) &= \mathrm{P}(\mathcal{X} \cap A \neq \varnothing) \\ &= \cdots \\ &= a_2^{1/2} - a_1\end{aligned}$$

- Hitting probability

$$\pi(x) = x^{1/2} - x$$



Hitting probability

Capacities, cont.

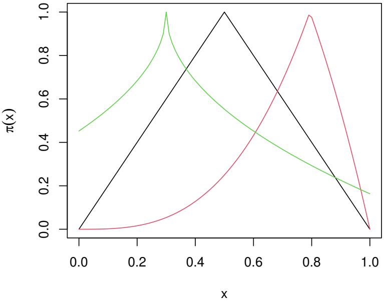
Example.[6] $\mathcal{X} = \{x \in \mathbb{X} : h(x) \geq U\}$, where is an $h : \mathbb{X} \to [0,1]$ *upper semi-continuous*[7] function, and U is a RV

- For a generic $A \subseteq \mathbb{X}$,

$$\begin{aligned} \overline{\Pi}(A) &= \mathrm{P}(\mathcal{X} \cap A \neq \varnothing) \\ &= \cdots \\ &= \mathrm{P}\left\{U \leq \sup_{x \in A} h(x)\right\} \end{aligned}$$

- Hitting probability

$$\pi(x) = \mathrm{P}\{U \leq h(x)\}$$



$\mathbb{X} = [0,1],\ U \sim \mathrm{Unif}(0,1)$

[6] Random upper level set

[7] $\limsup_{x_n \to x} h(x_n) \leq h(x)$: "$h(x) \geq u$ for some $x \in A$ iff $\sup_{x \in A} h(x) \geq u$"

Capacities, cont.

- First basic property: $\overline{\Pi}$ is 2-alternating[8]
- So, safe from a De Finetti-style sure loss criticism
- Actually, $\overline{\Pi}$ satisfies even stronger properties:
  - $\overline{\Pi}$ is ∞-monotone
  - $\overline{\Pi}$ is upper semi-continuous[9]
- Very regular class of capacities
- *Choquet's theorem.* The only capacities that satisfy the above properties are those corresponding to random sets

[8] equivalently, $\underline{\Pi}$ is 2-monotone

[9] that is, $\overline{\Pi}(K_n) \downarrow \overline{\Pi}(K)$ for compact $K_n \downarrow K$

Next lecture

- More on random sets & capacities
- Examples
- Choquet's theorem
- Maxitivity
- ...