ST790 — Fall 2022
Imprecise-Probabilistic Foundations of Statistics

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Week 03a

This lecture

- Part 1:
  - one more random set example
  - brief rant
- Part 2:
  - possibility theory
  - intuition and formal definition
  - first properties of possibility measures
  - examples

Example

- Common in robust statistics,[1] define the ε-*contamination class*

$$\mathscr{P} = \{(1-\varepsilon)\mathrm{P}_0 + \varepsilon\mathrm{Q} : \mathrm{Q} \text{ is any probability on } \mathbb{X}\}$$

  - $\varepsilon \in (0, 1)$ is a specified constant
  - P_0 is a specified probability on $\mathbb{X}$
- Define the *lower envelope* of $\mathscr{P}$,

$$\underline{\Pi}(A) = \inf_{\mathrm{P} \in \mathscr{P}} \mathrm{P}(A), \quad A \subseteq \mathbb{X}$$

- Claims:[2]
  1. $\underline{\Pi}$ is ∞-monotone (hence 2-monotone)
  2. $\mathscr{P} = \mathscr{C}(\underline{\Pi})$

[1] Huber, Kadane & Wasserman, Walley, ...
[2] Basically, Problem 3.6 in Nguyen's book

Brief rant

- What we're doing is bigger than just the math
- Recognition that *there's more to uncertainty than probability* is relevant to everyday life
- We're constantly being told what to do/think
- Rationale is as follows:
  - "ran the numbers" = fit a model, estimated probabilities
  - recommended policy based on maximum expected utility
  - it's "data-driven" and, therefore, "objective" and "right"
- Can complex, real-world problems be solved so precisely??

Brief rant, cont.

- *Naive Probabilism:*[3] Belief that all real-world decision problems can be solved as described in (prob) textbooks
- *Enlightened Probabilism:*

  When gambling, think probability
  When hedging, think plausibility
  When preparing, think possibility
  Otherwise, stop thinking — just survive

- Crane's presentation is more qualitative, but the connection to imprecise probability is clear
- The point is that precision is the outlier case, and that dealing with imprecision is key to critical thinking
- ST790 is about common sense and math

[3] Crane, `https://researchers.one/articles/21.04.00004`

Pause

Transition from Part 1 to Part 2

Introduction

- The capacities we've seen so far were *derived*, i.e., they inherited their properties from, say, a random set
- This is a convenient way to get started w/ imprecise prob
- But we could just directly define our class of capacities to have properties we want
- These properties would be driven by what we think *uncertainty quantification* ought to achieve
- Precise probabilities are too rigid
- Want more flexibility without too much added complexity

Intro, cont.

- *Basic insight:* probable $\implies$ possible, but not conversely
- Hence, *possibility* is more primitive than *probability*
- In statistics at least, often we're only asking for possibility-related conclusions — think hypothesis testing
- So, maybe it's enough to reason with possibility...
- *Possibility theory*[4] is a simple-yet-powerful framework for reasoning and uncertainty quantification
  - being simple means that it's not right for all applications
  - but I believe that it's "right" for statistics

[4] Closely related to fuzzy set theory; also to random sets & belief functions

Possibility

- Helps to think in terms of Shackle's *potential surprise*[5]
- Two extreme cases:
  - $\text{poss}(E) = 1 \rightarrow$ not surprised at all if E occurred
  - $\text{poss}(E) = 0 \rightarrow$ totally surprised if E occurred
- Degrees of possibility/surprise?
  - little surprised if it rained tomorrow
  - more surprised if it snowed tomorrow
  - totally surprised if sun didn't rise tomorrow
- *Possibility/surprise is not probability*[6]
  - multiple, mutually exclusive assertions can be 100% possible
  - e.g., I wouldn't be surprised if my grant proposal gets rejected, nor would I be surprised if it gets accepted

[5] This terminology makes clear that the notions of possibility we're talking about are subjective; "potential" indicates this is an advance assessment

[6] In fact, probability is *not* even a special case of possibility

Possibility, cont.

- To me, what distinguishes possibility & probability is this:

probability is relative, possibility is absolute

- So, roughly, possibility can be assessed separately for each E
- Example: draw a ball from a bag
  - poss(Green) need not depend on what's in the bag
  - equally-likely probability model,[7] say, needs to know how many distinct colors are in the bag
- The above statement is not strictly true — without some relationships we can't guarantee coherence
- That's where the math comes in...

[7] e.g., Laplace's *principle of insufficient reason*

Possibility, cont.

- Statistics: $X \sim \mathsf{P}_\theta$ with θ unknown
- None of the classical inference questions/objectives directly involve assigning *probabilities*[8] to θ
- All can be related to possibility/surprise, however:
  - point estimation $\to$ most possible parameter value?
  - hypothesis testing $\to$ is "H_0 true" sufficiently possible?
  - confidence sets $\to$ which θ's are (individually) suff. possible?
- Hypothesis tests:
  - Fisher's significance tests don't refer to an alternative
  - meant to be *absolute* assessments of whether data is sufficiently compatible with truthfulness of H_0
  - small p-value means I'd be sufficiently surprised if H_0 was true, doesn't say anything about, e.g., support for H_0^c

[8] *Falsificationist school* (Popper et al) says probabilities aren't needed

Example

- World is $\mathbb{X}$, true state is one of these values
- Suppose I learn that $E \subseteq \mathbb{X}$ contains the true state
- For uncertainty quantification about the unknown state, how might I assess the "possibility" of other assertions?
- Reasonable strategy:

$$\overline{\Pi}(A) = \begin{cases} 1 & \text{if } A \cap E \neq \varnothing \\ 0 & \text{otherwise,} \end{cases} \quad A \subseteq \mathbb{X}$$

- i.e., if $A \cap E \neq \varnothing$, then wouldn't be surprised if A were true; if $A \cap E = \varnothing$, then totally surprised if A were true
- *Total ignorance* is the special case $E = \mathbb{X}$

Example, cont.

- Note that $\overline{\Pi}$ above clearly satisfies

$$\overline{\Pi}(A \cup B) = \max\{\overline{\Pi}(A), \overline{\Pi}(B)\}, \quad \text{all } A, B \subseteq \mathbb{X}$$

- This is the *maxitivity* property from before
- Therefore, $\overline{\Pi}$ is very nice/simple:
  - $\overline{\Pi}$ is ∞-alternating
  - induced by a random set (what is it?)
- Upon careful reflection, maxitivity is the only way for possibility to be "absolute" in the sense above
- So, maxitivity is taken as an *axiom* of possibility theory

Possibility measures

Definition

A functional $\overline{\Pi} : 2^{\mathbb{X}} \to [0,1]$ is a *possibility measure* if

1. $\overline{\Pi}(\varnothing) = 0$
2. $\overline{\Pi}(\mathbb{X}) = 1$
3. $\overline{\Pi}$ is maxitive

The conjugate/dual, $\underline{\Pi}$, of $\overline{\Pi}$ is a *necessity measure*

- Like usual, $\underline{\Pi}(\cdot) \leq \overline{\Pi}(\cdot)$
- Unique feature of possibility: inequality is "basically strict"

$$\underline{\Pi}(A) > 0 \implies \overline{\Pi}(A) = 1$$
$$\overline{\Pi}(A) < 1 \implies \underline{\Pi}(A) = 0$$

Possibility measures, cont.

- As with general maxitive capacities, everything is determined by a "hitting probability" function
- Called a *possibility distribution* or *possibility contour*[9]
- Starting with a usc function $\pi : \mathbb{X} \to [0,1]$ that satisfies the property $\sup_{x \in \mathbb{X}} \pi(x) = 1$, define

$$\overline{\Pi}(A) = \sup_{x \in A} \pi(x), \quad A \subseteq \mathbb{X}$$

- $\overline{\Pi}$ is a possibility, and the corresponding necessity is

$$\underline{\Pi}(A) = 1 - \overline{\Pi}(A^c) = 1 - \sup_{x \in A^c} \pi(x), \quad A \subseteq \mathbb{X}$$

[9] I may say "plausibility" instead of "possibility" for reasons that'll be clear...

Example

- Let $\mathbb{X} = [0, 1]$ and let F be a CDF[10] on $\mathbb{X}$
- Define the function $\pi(x) = 1 - |2F(x) - 1|$
- Possibility and necessity:

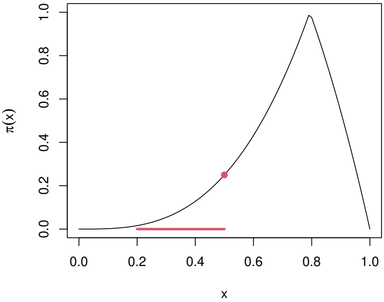
$$\overline{\Pi}(A) = \sup_{x \in A} \pi(x)$$

$$\underline{\Pi}(A) = 1 - \sup_{x \in A^c} \pi(x)$$

- Induced by the random set

$$\mathcal{X} = \{x : \pi(x) \geq \pi(X)\}$$

where $X \sim F$



[10] Plot is for $F = \text{Beta}(3, 1)$

Next lecture

- More possibility theory
- Examples
- Properties
- Connections to imprecise probability