ST790 — Fall 2022

Imprecise-Probabilistic Foundations of Statistics

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Week 03b

This lecture[1]

- Possibility measures & basic properties
- Examples
- Imprecise probability aspects
  - credal set contents
  - no sure loss
  - coherence (?)

[1] Most of what I'm presenting here is taken from Dominik Hose's 2022 PhD thesis (University of Stuttgart). I was a member of his thesis committee so I'm most familiar with his presentation very clear presentation of (what I think are) the most relevant basics of possibility theory

Possibility measures

- Recall, a function $\overline{\Pi} : 2^{\mathbb{X}} \to [0,1]$ is a *possibility measure* if
  - $\overline{\Pi}(\varnothing) = 0$
  - $\overline{\Pi}(\mathbb{X}) = 1$
  - it's maxitive, i.e., $\overline{\Pi}(\bigcup_{n=1}^{\infty} A_n) = \sup_n \overline{\Pi}(A_n)$
- Consequently, there exists a function $\pi : \mathbb{X} \to [0,1]$, called the *possibility contour*, such that $\sup_{x \in \mathbb{X}} \pi(x) = 1$ and

$$\overline{\Pi}(A) = \sup_{x \in A} \pi(x), \quad A \subseteq \mathbb{X}$$

- The dual, $\underline{\Pi}$, is a *necessity measure* and satisfies

$$\underline{\Pi}(A) = 1 - \overline{\Pi}(A^c) = 1 - \sup_{x \in A^c} \pi(x), \quad A \subseteq \mathbb{X}$$

Example

- Let $\mathbb{X} = [0, 1]$ and let F be a CDF[2] on $\mathbb{X}$
- Define the function $\pi(x) = 1 - |2F(x) - 1|$
- Possibility and necessity:

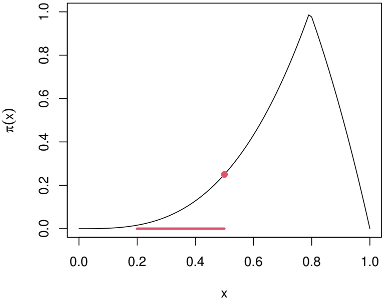
$$\overline{\Pi}(A) = \sup_{x \in A} \pi(x)$$

$$\underline{\Pi}(A) = 1 - \sup_{x \in A^c} \pi(x)$$

- Induced by the random set

$$\mathcal{X} = \{x : \pi(x) \geq \pi(X)\}$$

where $X \sim F$



[2] Plot is for $F = \text{Beta}(3, 1)$

Possibility measures, cont.

- An advantage of possibility theory is that it's simple, arguably the simplest of IP models
- The reason it's simple: $\overline{\Pi}$ is determined by π
- Compare to probability:
  - everything is done with probability density/mass function
  - difference is the calculus, optimization vs. integration
- Close connections to p-values and hypothesis tests; even more connections to statistics later
- Other "imprecise probability" properties...?

Possibility measures, cont.

- Let $\overline{\Pi}$ be a possibility measure with contour π
- Some definitions:
  - $\{x : \pi(x) > 0\}$ is the *support* of π (or $\overline{\Pi}$)
  - $\{x : \pi(x) = 1\}$ is the *core* of π (or $\overline{\Pi}$)
- For $\alpha \in [0, 1]$, define *super-* and *sub-level* sets[3]

$$S_\alpha(\pi) = \{x : \pi(x) > \alpha\}$$
$$S_\alpha^c(\pi) = \{x : \pi(x) \leq \alpha\}$$

- Clearly, these sets are *nested*, e.g.,

$$\alpha \leq \beta \implies S_\alpha(\pi) \supseteq S_\beta(\pi) \text{ and } S_\alpha^c(\pi) \subseteq S_\beta^c(\pi)$$

[3] Let "*S*" stand for "super"...

Possibility measures, cont.

- Level sets are fundamental to $\overline{\Pi}$
- Two observations:

$$\overline{\Pi}\{S_\alpha^c(\pi)\} \leq \alpha \quad \text{and} \quad \underline{\Pi}\{S_\alpha(\pi)\} \geq 1 - \alpha$$

- Superficial similarity to "coverage probability" of CIs...
- They also basically determine possibility of other events
  - for $A \subseteq \mathbb{X}$, let $\alpha(A) = \inf\{\alpha : S_\alpha^c(\pi) \supseteq A\}$
  - then $S_{\alpha(A)}^c(\pi)$ is the "smallest sublevel set containing A"
  - and $\overline{\Pi}(A) = \alpha(A)$
- Sketch a picture...

Credal set contents

- *Question:* If $\overline{\Pi}$ is a possibility measure, then what probabilities are contained in the credal set $\mathscr{C}(\overline{\Pi}) = \{\mathsf{P} : \mathsf{P} \leq \overline{\Pi}\}$?
- That is, can we characterize those $\mathsf{P} \in \mathscr{C}(\overline{\Pi})$?
- In particular:
  1. is $\mathscr{C}(\overline{\Pi}) \neq \varnothing$? .... (no-sure-loss)
  2. is $\overline{\Pi}(\cdot) = \sup_{\mathsf{P} \in \mathscr{C}(\overline{\Pi})} \mathsf{P}(\cdot)$? .... (coherence)

Credal set contents, cont.

Theorem.

For a given $\overline{\Pi}$ with contour π, let $S_\alpha = S_\alpha(\pi)$ be the super-level sets. Then $\mathsf{P} \in \mathscr{C}(\overline{\Pi})$ iff $\mathsf{P}(S_\alpha) \geq 1 - \alpha$ for all $\alpha \in [0, 1]$

- That is, P is *consistent* with $\overline{\Pi}$ iff it assigns mass $\geq 1 - \alpha$ to the α-super-level sets of $\overline{\Pi}$
- Equivalent to check that $\mathsf{P}(S_\alpha^c) \leq \alpha$ for all α

Credal set contents, cont.

Proof.

This is an *if and only if* so there's two implications to prove.

- Since $\overline{\Pi}(S_\alpha^c) \leq \alpha$, if $\mathsf{P} \leq \overline{\Pi}$, then $\mathsf{P}(S_\alpha^c) \leq \alpha$.
- Next, suppose P is such that $\mathsf{P}(S_\alpha^c) \leq \alpha$ for all α. Take any A and set $\beta = \overline{\Pi}(A)$. Then $A \subseteq S_\beta^c$ and, therefore,

$$\mathsf{P}(A) \leq \mathsf{P}(S_\beta^c) \leq \beta = \overline{\Pi}(A).$$

$\square$

Credal set contents, cont.

- $\mathscr{C}(\overline{\Pi}) \neq \varnothing$, hence *no-sure-loss*, if there's one $\mathsf{P} \in \mathscr{C}(\overline{\Pi})$
- One case is obvious:
  - suppose $x^\star \in \text{core}(\pi)$ is in the interior[4] of $\mathbb{X}$
  - then $\delta_{x^\star} = [\text{point mass at } x^\star] \in \mathscr{C}(\overline{\Pi})$
- A more general construction is as follows:
  - Take any P_0 that assigns non-zero mass to S_α's
  - Define a new probability measure

$$\mathsf{P}^\star(A) = \int_0^1 \frac{\mathsf{P}_0(S_\alpha \cap A)}{\mathsf{P}_0(S_\alpha)} \, d\alpha, \quad \text{for } \mathsf{P}_0\text{-measurable } A$$

  - Then $\mathsf{P}^\star \in \mathscr{C}(\overline{\Pi})$

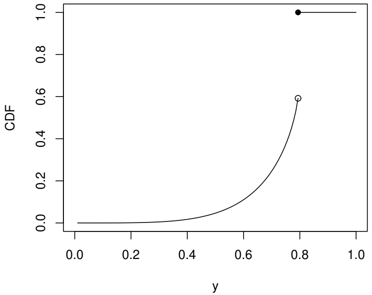
[4] This doesn't work if the core is "at ∞"...

Example, cont.

- $\pi(x) = 1 - |2F(x) - 1|$ where F is a CDF on $\mathbb{X} = [0, 1]$
- Core is the median of F, super-level sets are

$$S_\alpha = \{x : \pi(x) > \alpha\} = \{x : \tfrac{\alpha}{2} < F(x) < 1 - \tfrac{\alpha}{2}\}$$

- Plot: CDF of $\mathrm{P}^\star$ with $F = \mathrm{Beta}(3, 1)$ and $\mathrm{P}_0 = \mathrm{Unif}(0, 1)$



Credal set contents, cont.

- Does $\overline{\Pi}(\cdot) = \sup_{P \in \mathscr{C}(\overline{\Pi})} P(\cdot)$?
- Of course, it's clear that $\overline{\Pi}(\cdot) \leq \sup_{P \in \mathscr{C}(\overline{\Pi})} P(\cdot)$
- That equality is achieved means $\overline{\Pi}$ is a tight upper bound and, therefore, that $\overline{\Pi}$ is coherent
- Very general proofs of coherence for possibility measures:
  - De Cooman & Aeyels (1999)
  - Bronevich & Rozenberg (2020)
- More elementary proof in Hose's thesis...[5]

[5] Hose attributes his proof to Fetz & Oberguggenberger (2004)

Next lecture

- Finish up details of coherence
- (Imprecise-)probability-to-possibility transform
- Extension principle
- ...