ST790 — Fall 2022
Imprecise-Probabilistic Foundations of Statistics

Ryan Martin
North Carolina State University
`www4.stat.ncsu.edu/~rmartin`

Week 05b

This lecture

- Credal sets and lower/upper envelopes
- no-sure-loss and coherence
- Choquet integration

Introduction

- Covered several different kinds of *IP models*
  - random sets
  - possibility measures
  - belief functions
- All are related, all relatively simple
- In particular, all are ∞-monotone and, therefore, (easily) meet the no-sure-loss/coherence requirements
- Next step: more general/flexible/complex models
- Iron out some technical details first...

- Notation: write $\overline{\mathsf{P}}$ (instead of $\overline{\Pi}$) for upper prob on $\mathbb{X}$
- Our focus so far:
  - given, say, an upper probability $\overline{\mathsf{P}}$
  - asked about the contents of its credal set, $\mathscr{C}(\overline{\mathsf{P}})$
- For example:
  - $\mathscr{C}(\text{poss})$ described via contour's level sets
  - $\mathscr{C}(\text{plaus})$ described via allocations
- More generally: under what conditions on $\overline{\mathsf{P}}$ can we say
  - $\mathscr{C}(\overline{\mathsf{P}}) \neq \varnothing$?
  - $\overline{\mathsf{P}}(A) = \sup_{\mathsf{P} \in \mathscr{C}(\overline{\mathsf{P}})} \mathsf{P}(A)$ for all A?[1]

[1] This sup is often called the *upper envelope*, inf for *lower envelope*

Intro, cont.

- Latter questions related to *no-sure-loss* and *coherence*
- I side-stepped these details so far — for good reason (?)
- Now it's time to address them...
- Connects to notions of lower/upper expectation
- Needed to motivate & understand the next topic

- To avoid non-trivial technical details, assume $\mathbb{X}$ is finite

Example

- Not all $(\underline{\mathrm{P}}, \overline{\mathrm{P}})$ pairs avoid sure loss or are coherent!
- Based on Example 10.1 in Huber & Ronchetti's book:[2]

$\lvert A \rvert$	0	1	2	3	4
$\underline{\mathrm{P}}(A)$	0	0	0.5	0.5	1
$\overline{\mathrm{P}}(A)$	0	X	0.5	1	1

- $(\underline{\mathrm{P}}, \overline{\mathrm{P}})$ are capacities for a range of X values
- But note that the $\lvert A \rvert = 2$ case imposes strong restrictions
- In particular, there's at most one P in $\mathscr{C}(\overline{\mathrm{P}})$
- That is, $\mathscr{C}(\overline{\mathrm{P}})$ equals $\varnothing$ or $\{\mathrm{Unif}(\mathbb{X})\}$

[2] $\lvert \mathbb{X} \rvert = 4$; $\underline{\mathrm{P}}(A)$ & $\overline{\mathrm{P}}(A)$ only depend on $\lvert A \rvert$

Example, cont.

- Suppose X is 0.2
  - Unif($\mathbb{X}$) not compatible with this upper bound
  - so $\mathscr{C}(\overline{\mathrm{P}}) = \varnothing$
  - and you can make me a *sure loser*...

    I'm willing to sell you \$1 bets on all four events A, with $|A| = 1$, for \$0.20 each; I get \$0.80 but you always win \$1
- Suppose X is 0.5
  - Unif($\mathbb{X}$) is compatible with this upper bound
  - so $\mathscr{C}(\overline{\mathrm{P}}) = \{\mathrm{Unif}(\mathbb{X})\}$
  - but silly to interpret $\overline{\mathrm{P}}$ as an "upper probability"

No sure loss

Theorem.

$\overline{\mathrm{P}}$ is such that $\mathscr{C}(\overline{\mathrm{P}}) \neq \varnothing$ iff

$$\sup_x \sum_{i=1}^n a_i\{\overline{\mathrm{P}}(A_i) - 1_{A_i}(x)\} \geq 0, \quad \text{all } n, \text{ all sets } A_i, \text{ all } a_i \geq 0$$

- What's the intuition?
  - I'm willing to accept $\overline{\mathrm{P}}(A_i)$ for a \$1 bet on A_i
  - i.e., I "expect" $a_i\{\overline{\mathrm{P}}(A_i) - 1_{A_i}(x)\}$ to be positive
  - if above fails, then there's a sequence of transactions that I'd accept that are sure to make my net negative
- In most texts, the condition of the above theorem is taken as the *definition* of $\overline{\mathrm{P}}$ avoiding sure loss

Coherence

Theorem.

$\overline{\mathrm{P}}$ equals the upper envelope of $\mathscr{C}(\overline{\mathrm{P}})$ iff

$$\sup_x \Big[\sum_{i=1}^{n} a_i\{\overline{\mathrm{P}}(A_i) - 1_{A_i}(x)\} - a_0\{\overline{\mathrm{P}}(A_0) - 1_{A_0}(x)\}\Big] \geq 0,$$

for all n, all sets A_i, all $a_i \geq 0$

- What's the intuition?
  - if condition fails (for $a_0 > 0$), then, for some $\delta > 0$ and all x

    $$\textstyle\sum_{i=1}^{n} a_i\{\overline{\mathrm{P}}(A_i) - 1_{A_i}(x)\} \leq a_0\{\overline{\mathrm{P}}(A_0) - 1_{A_0}(x)\} - \delta$$

  - ... so $\overline{\mathrm{P}}(A_0)$ is too high, not a tight upper prob
- In most texts, the condition of the above theorem is taken as the *definition* of $\overline{\mathrm{P}}$ being coherent

Achieving coherence

- A while back I said in class that $\overline{P}$ is coherent if it's 2-alternating, equivalently, if $\underline{P}$ is 2-monotone
- There's a round-about way of verifying this, which I'll only give a very rough sketch of
- "Round-about" in the sense that it takes a route through *lower* and *upper expectations*
  - precise case: probs $\Longleftrightarrow$ expectations
  - imprecise case: expectations are more expressive
- It's for this reason that, outside the cases we've considered so far, the focus is on expectations

Choquet integral

- A capacity is like a measure — if we can integrate wrt measures, then can't we do the same wrt capacities?
- Decision-making: $\overline{\mathrm{P}}$ quantifies my uncertainty, so might want to choose action to maximize "upper expected utility"
- The *Choquet integral* generalizes the Lebesgue integral
- Recall from probability theory:
  - If f is non-negative, then

$$\mathrm{E}f := \mathrm{E}f(X) = \int_0^\infty \mathrm{P}\{f(X) > s\}\, ds$$

  - Extend to general f by splitting as $f = f^+ - f^-$...

Choquet integral, cont.

Definition.

Let $\overline{\mathrm{P}}$ be a capacity on $\mathbb{X}$ and $f : \mathbb{X} \to \mathbb{R}$ a function. Then the *Choquet integral* of f with respect to $\overline{\mathrm{P}}$ is

$$\overline{\mathrm{E}}f = \int_0^\infty \overline{\mathrm{P}}\{x : f(x) > s\}\, ds - \underbrace{\int_{-\infty}^0 [1 - \overline{\mathrm{P}}\{x : f(x) > s\}]\, ds}_{= 0 \text{ for non-negative } f}$$

- Integrals on the RHS are of the Riemann variety
- Clearly a generalization of the Lebesgue integral
- Not linear like familiar integrals
- Get $\underline{\mathrm{E}}f$ using same formula with $\underline{\mathrm{P}}$ in place of $\overline{\mathrm{P}}$

Choquet integral, cont.

- Possibility measure (general $\mathbb{X}$)
  - let $\overline{\mathrm{P}}$ be a possibility measure with contour p
  - Choquet integral of $f \geq 0$:

$$\begin{aligned}\overline{\mathrm{E}}f &= \int_0^\infty \overline{\mathrm{P}}\{x : f(x) > s\}\,ds \\ &= \inf f + \int_{\inf f}^{\sup f} \left\{ \sup_{x: f(x) > s} p(x) \right\} ds = \underbrace{\int_0^1 \left\{ \sup_{x: p(x) > \alpha} f(x) \right\} d\alpha}_{\text{not immediate!}}\end{aligned}$$

- Belief function (finite $\mathbb{X}$):
  - $\underline{\mathrm{P}}$ is a belief function with mass m
  - Choquet integral of $f \geq 0$:

$$\underline{\mathrm{E}}f = \int_0^\infty \underline{\mathrm{P}}\{x : f(x) > s\}\,ds = \sum_A \left\{ \min_{x \in A} f(x) \right\} m(A)$$

More coherence

- Generic set of probabilities $\mathscr{P}$ on $\mathbb{X}$
- Define lower and upper envelopes (expectations)

$$\underline{\mathrm{E}}f = \inf_{\mathrm{P}\in\mathscr{P}} \mathrm{E}f \quad \text{and} \quad \overline{\mathrm{E}}f = \sup_{\mathrm{P}\in\mathscr{P}} \mathrm{E}f, \quad f : \mathbb{X} \to \mathbb{R}$$

- $\mathscr{C}(\overline{\mathrm{E}}) := \{\mathrm{P} : \mathrm{E}f \leq \overline{\mathrm{E}}f \text{ for all } f\}$
- (More) natural to talk about coherence with expectations

Coherence theorem.

1. Let $\mathscr{P}$ be given and define $\overline{\mathrm{E}}$ as the upper envelope. Then $\mathscr{P} = \mathscr{C}(\overline{\mathrm{E}})$ iff $\mathscr{P}$ is closed and convex
2. Let $\overline{\mathrm{E}}$ be given. Then $\overline{\mathrm{E}}$ is the upper envelope of $\mathscr{C}(\overline{\mathrm{E}})$ iff
   - $f \leq g$ implies $\overline{\mathrm{E}}f \leq \overline{\mathrm{E}}g$
   - $\overline{\mathrm{E}}(af + b) = a\overline{\mathrm{E}}f + b$, for all $(a, b) \in \mathbb{R}_+ \times \mathbb{R}$
   - $\overline{\mathrm{E}}(f + g) \leq \overline{\mathrm{E}}f + \overline{\mathrm{E}}g$

- *Claim:* If $\overline{\mathrm{P}}$ is 2-alternating, then it's coherent

- Choquet integral $\overline{\mathrm{E}}$ of $\overline{\mathrm{P}}$ satisfies Part 2 of above theorem
- So, it's the upper envelope of its induced credal set
- Hence, coherence

Next lecture

- De Finetti's gambles and previsions
- Coherent lower/upper previsions
- Examples
- Properties