ST790 – Homework 4 Due: 11/22/2022

These exercises are meant to supplement the lectures by providing some further examples to illustrate the general ideas and theory. Students (individually or in pairs) should attempt to solve all the assigned problems. The solutions will be collected at the end of the semester (the day before Thanksgiving), so you may work on these at your own pace. But don't wait too long to get started! If you have questions, feel free to ask.

1. (Example 5.13.11 on page 280 of Walley's book.) Let X be an uncertain integer between 1 and 99; it helps to think of the "single-digit" numbers as 01, 02, ..., 09. You know that the first digit in X is randomly assigned a value 0–9 but all you know about the second digit is that it's non-zero. You opt to quantify your uncertainty about X using a belief function with focal elements

$$A_j = \{10j+1, 10j+2, \ldots, 10j+9\}, \quad j = 0, 1, \ldots, 9,$$

and mass function satisfying $m(A_j) = 0.1$, for $j = 0, 1, \ldots, 9$.

Define a partition of the X-space as

$$B_k = \{0k\} \cup A_k, \quad k = 1, \ldots, 9.$$

Observing "$X \in B_k$" amounts to learning that the first non-zero digit in X is k. In other words, "$X \in B_k$" means that X is of the form $k\_$ or $0k$.

   (a) The goal is to update the original beliefs, based on observation B_k, using Dempster's rule of combination/conditioning.

      i. Argue that only two of the original focal elements, namely, A_0 and A_k, are compatible (have non-empty intersection) with B_k.

      ii. Then Dempster's rule amounts to renormalizing the original masses of these two compatible focal elements. In that case, show that the updated belief function has mass $m(\cdot \mid B_k)$ that satisfies

$$m(\{0k\} \mid B_k) = m(A_k \mid B_k) = 0.5.$$

   (b) Find the definition of *sure loss* in Gong & Meng (*Stat Sci* 2021).[1] Use the result in Part (a) above to argue that Dempster's rule incurs sure loss.

   (c) Explain in what sense this is a "sure loss." That is, if your original buying and selling prices for gambles are determined by the belief function above, and you update them according to Dempster's rule as described in Part (a), then describe the strategy that an opponent can take to make you a sure loser.

   Hint: How does the belief assigned to A_0 change after B_k is observed?

2. The false confidence phenomenon, while always present, can be hard to detect. The reason is that, in simple one-dimensional problems, the assertions afflicted by false confidence typically aren't the ones we'd naturally consider. In higher-dimensional cases, there are so many candidate assertions to consider but the details are more

[1] Link on the course website, under Week 07a.

complicated and less transparent. To see the phenomenon most clearly, without making the example too complex, it helps to look at "weird" situations.[2] One relatively simple example where false confidence is not too difficult to see is in the ratio-of-normal-means problem.

(a) Reproduce the simulation results presented in Figure 3 in RM's 2021 paper.[3] You can use 1000 samples from the posterior/confidence distribution of (θ_1, θ_2) and 1000 replications to get the distribution function curve.

(b) Redo the above experiment with different true values of (θ_1, θ_2) so that the false confidence phenomenon you observed in Part (a) disappears or at least becomes less severe.

Hint: Part of what makes this example "weird" is that the mean in the denominator is allowed to be close/equal to 0. If the true value in the denominator isn't close to 0, then this example isn't as "weird."

3. Consider a Poisson model where $(Y \mid \Theta = \theta) \sim \mathsf{P}_{Y|\theta} = \mathsf{Pois}(\theta)$; assume the prior information about Θ is vacuous. Dempster (*IJAR* 2008)[4] describes his approach to making inference on Θ based on the observation $Y = y$.

   (a) Verify the expression in Equation (6) of Dempster's paper.

   Hint: Notation and terminology aside, this boils down to a property concerning a pair of dependent random variables (T_k, T_{k+1}), defined by $T_k = \sum_{i=1}^k U_i$ and $T_{k+1} = T_k + U_{k+1}$, where $U_1, U_2, \ldots$ are iid $\mathsf{Exp}(1)$ random variables.

   (b) Using my notation from class, Dempster's random set for inference on Θ, given $Y = y$, is the random interval

   $$\mathcal{T}_y(U) = [T_y, T_{y+1}], \quad U = (U_1, U_2, \ldots),$$

   where the U's—which determine the T's—are as defined in Part (a) above. Use the formula from Equation (6) in Dempster's paper, as derived above, to find an expression for the hitting probability function

   $$\pi_y(\theta) := \mathsf{P}_U\{\mathcal{T}_y(U) \ni \theta\}, \quad \theta > 0.$$

   Draw a plot of this function when $y = 7$ is the observed value.

   (c) In the vacuous prior case, validity and strong validity are equivalent. In other words, Dempster's IM would be valid if and only if

   $$\mathsf{P}_{Y|\theta}\{\pi_Y(\theta) \le \alpha\} \le \alpha, \quad \text{for all } \alpha \in [0,1] \text{ and all } \theta > 0.$$

   Do a simulation to approximate the distribution function of $\pi_Y(\theta)$, as a function of $Y \sim \mathsf{Pois}(\theta)$, for a few different θ values. Do you think Dempster's IM is valid? Why or why not?

[2] That it helps to look at "weird" examples shouldn't be surprising. Good tests of a subject's foundations are never the standard or straightforward examples. For example, issues with the Newtonian theory of mechanics aren't apparent except at extremely large or small scales.

[3] `https://researchers.one/articles/21.01.00002`

[4] Link on the course website, under Week 08b.

4. Reconsider the Poisson example above but now refer to the generalized Bayes-based IM constructed in Walley (*JSPI* 2002).[5] His IM has a contour function, which I'll denote here as $\pi_y(\theta)$, defined in Equation (3.3) of the paper, which can be applied to any statistical model, in particular, the Poisson model.

   (a) Using Walley's notation, let Q denote a $\mathsf{Gamma}(a, b)$ distribution, where $a > 0$ is the shape and $b > 0$ is the scale. Write out the expression for $\pi_y(\theta)$, which will depend on (y, ε, a, b).

   Hint: Recall that gamma priors are conjugate to Poisson likelihoods.

   (b) Draw a plot of the contour function for $y = 7$, $\varepsilon = 0.7$, $a = 0.01$ and $b = 100$.

   (c) Repeat the simulation in Problem 3(c) but for Walley's IM. Does validity hold?

5. An IM construction for the Poisson model is described in Section 2 of M. and Liu (*JASA* 2013).[6] The contour function is defined in Equation (2.13).

   (a) Draw a plot of the contour function based on $y = 7$.

   (b) Repeat the above simulation with this IM construction. Validity?

6. Follow the recipe described in the Week 10a lecture to construct a strongly valid, likelihood-based IM for the Poisson model with vacuous prior information. Don't forget the dimension reduction step explained on Slides 12–13.

   (a) Plot the contour function when $y = 7$. How does this contour plot compare to those of the other IMs in the previous problems?

   (b) Repeat the above simulation and confirm that strong validity holds.

[5] Link on the course website, under Week 06b.

[6] `https://arxiv.org/abs/1206.4091`